

# The power of two choices in distributed voting\*

Colin Cooper<sup>†</sup>Robert Elsässer<sup>‡</sup>Tomasz Radzik<sup>§</sup>

April 30, 2014

## Abstract

Distributed voting is a fundamental topic in distributed computing. In the standard model of pull voting, in each step every vertex chooses a neighbour uniformly at random, and adopts the opinion of that neighbour. The voting is said to be completed when all vertices hold the same opinion. On many graph classes including regular graphs, irrespective of the expansion properties, pull voting requires  $\Omega(n)$  expected time steps to complete, even if initially there are only two distinct opinions with the minority opinion being sufficiently large.

In this paper we consider a related process which we call two-sample voting. In this process every vertex chooses two random neighbors in each step. If the opinions of these neighbors coincide, then the vertex revises its opinion according to the chosen sample. Otherwise, it keeps its own opinion. We consider the performance of this process in the case where two different opinions reside on vertices of some (arbitrary) sets  $A$  and  $B$ , respectively. Here,  $|A| + |B| = n$  is the number of vertices of the graph.

We show that there is a constant  $K$  such that if the initial imbalance between the two opinions is  $\nu_0 = (|A| - |B|)/n \geq K\sqrt{(1/d) + (d/n)}$ , then with high probability two sample voting completes in a random  $d$  regular graph in  $O(\log n)$  steps and the initial majority opinion wins. We also show the same performance for any regular graph, if  $\nu_0 \geq K\lambda_2$ , where  $\lambda_2$  is the second largest eigenvalue of the transition matrix. In the graphs we consider, standard pull voting requires  $\Omega(n)$  steps, and the minority can still win with probability  $|B|/n$ . Our results hold even if an adversary is able to rearrange the opinions in each step, and has complete knowledge of the graph structure.

## 1 Introduction

Distributed voting has applications in various fields including consensus and leader election in large networks [6, 19], serialisation of read/write in replicated data-bases [18], and the analysis of social behaviour in game theory [12]. Voting algorithms are usually simple, fault-tolerant, and easy to implement [19, 21].

One straightforward form of distributed voting is *pull voting*. In the beginning each vertex of a connected undirected graph  $G = (V, E)$  has an initial opinion. The voting process proceeds synchronously in discrete time steps called rounds. During each round, each vertex independently contacts a random neighbour and adopts the opinion of that neighbour. The completion

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\*This work was partially supported by EPSRC grant EP/J006300/1, “Random Walks on Computer Networks”, the Austrian Science Fund (FWF) under contract P25214-N23 “Analysis of Epidemic Processes and Algorithms in Large Networks”, and the 2012 SAMSUNG Global Research Outreach (GRO) grant “Fast Low Cost Methods to Learn Structure of Large Networks.”

<sup>†</sup>Department of Informatics, King’s College London, UK. [colin.cooper@kcl.ac.uk](mailto:colin.cooper@kcl.ac.uk)

<sup>‡</sup>Department of Computer Sciences, University of Salzburg, Austria. [elsa@cosy.sbg.ac.at](mailto:elsa@cosy.sbg.ac.at)

<sup>§</sup>Department of Informatics, King’s College London, UK. [tomasz.radzik@kcl.ac.uk](mailto:tomasz.radzik@kcl.ac.uk)

time  $T$  is the number of rounds needed for a single opinion to emerge. This time depends on the structure of the underlying graph, and is normally measured in terms of its expectation  $\mathbf{E}T$ . We showed in [8] that with high probability the completion time is  $O(n/(\nu(1 - \lambda_2)))$ , where  $n$  is the number of vertices,  $\lambda_2$  is the second largest eigenvalue of the transition matrix,  $\nu = \sum_{v \in V} d^2(v)/(d^2 n)$ ,  $d(v)$  is the degree of vertex  $v$  and  $d$  is the average degree.

In the *two-party voter model*, vertices initially hold one of two opinions  $A$  and  $B$ . As usual, the pull voting is completed when all vertices have the same opinion. Hassin and Peleg [19] and Nakata *et al.* [23] considered the discrete-time two-party voter model on connected graphs, and discussed its application to consensus problems in distributed systems. Both papers focus on analysing the probability that all vertices will eventually adopt the opinion which is initially held by a given group of vertices.

Let  $A$  and  $B$  denote also the sets of vertices with opinions  $A$  and  $B$ , respectively;  $A \cup B = V$ . Let  $d(X)$  be the sum of the degrees of the vertices in a set  $X$ . We say that opinion  $A$  wins, if all vertices eventually adopt this opinion. The central result of [19] and [23] is that the probability that opinion  $A$  wins is

$$P_A = \frac{d(A)}{2m}, \quad (1)$$

where  $m$  is the number of edges in  $G$ . Thus in the case of connected regular graphs, the probability that  $A$  wins is proportional to the original size of  $A$ , irrespective of the graph structure. Apart from the probability of winning the vote, another quantity of interest is the time  $T$  taken for voting to complete. In [19] it is proven that  $\mathbf{E}T = O(n^3 \log n)$  for general connected  $n$  vertex graphs. For the case of random  $d$ -regular graphs, it is shown in [9] that  $\mathbf{E}T \sim 2n(d-1)/(d-2)$  with high probability. It follows from the proof of this result that, with high probability, two-party voting needs  $\Theta(n)$  time to complete on random  $d$ -regular graphs.

The performance of the two-party pull-voting seems unsatisfactory in two ways. Firstly, it is reasonable to require that a clear majority opinion wins with high probability. From (1) it follows that, even if initially only a single vertex  $v$  holds opinion  $A$ , then this opinion wins with probability  $P_A = d(v)/2m$ . Secondly, the expected completion time is at least  $\Omega(n)$  on many classes of graphs, including regular expanders and complete graphs. This seems a long time to wait to resolve a dispute between two opinions. A more reasonable waiting time would depend on the graph diameter, which is  $O(\log n)$  for many important classes of graphs including expanders.

To address these issues, we consider a modified version of pull voting in which each vertex  $v$  randomly queries two neighbours at each step. On the basis of the sample taken, vertex  $v$  revises its opinion as follows. If both neighbours have the same opinion, the calling vertex  $v$  adopts this opinion. If the two opinions differ, the calling vertex  $v$  retains its current opinion in this round. To distinguish this process from the conventional pull voting, as described above, we use terms *single-sample voting* and *two-sample voting*. The aim of the two-sample voting is to ensure that voting finishes quickly and the initial majority opinion wins (almost always). The two-sample voting is intrinsically attractive, as it seems to mirror the way people behave. If you hear it twice it must be true.

In [7] we analysed a two sample process called min-voting. Here, initially each vertex holds a distinct opinion. In each step every vertex chooses two neighbours uniformly at random and takes the smaller opinion of the two. For graphs with good expansion properties we proved that min-voting completes in time  $O(\log n)$ , with high probability. Although min-voting is fast, an adversary with somewhat limited abilities could break the system by continuously introducing small numbers into the network. Moreover the model is meaningless in two party voting, as the smaller opinion always wins.

In this paper we analyse two-sample voting for two classes of  $d$ -regular graphs: random graphs and expanders parameterized by the eigenvalue gap. Our results depend only on the initial imbalance  $\nu_0 = (|A| - |B|)/n$ . As an example, for random  $d$ -regular graphs there is an absolute constant  $K$ , independent of  $d$ , such that provided

$$\nu_0 \geq K \sqrt{\frac{d}{n} + \frac{1}{d}},$$

with high probability two-sample voting is completed in  $O(\log n)$  steps and the winner is the opinion with the initial majority. We discuss our results in more detail in the next section. The main advantages of our two-sample voting are that the completion time speeds up from  $\Theta(n)$  to  $O(\log n)$  and with high probability the initial majority opinion wins.

It seems interesting to enquire further how the performance of pull voting systems depends on the range of choices available in the design. We restrict our discussion to two-party voting. The main issues seem to be the number of neighbours  $k$  to contact at each step, and the rule used to reach a decision based on the opinions obtained. In the case  $k = 1$ , this is single-sample pull voting, as discussed above. For  $k = 2$ , a simple rule is to adopt the opinion if both neighbours agree (the voting protocol analysed in this paper). For  $k \geq 3$  odd, a comparable rule is to adopt the majority opinion. Interestingly, the number  $k$  of neighbours contacted at each step can substantially influence the performance of the process in at least three ways: the completion time, the final outcome, and the robustness of the system against adversarial attacks.

We briefly compare the performance of such systems for two-party voting on random  $d$ -regular graphs for various values of  $k$ . Surprisingly, a clearly defined complexity hierarchy emerges, which distinguishes between  $k = 1$ ,  $k = 2$  and  $k \geq 5$  odd.

- $k = 1$ . As previously mentioned, the expected completion time in this case is  $\Theta(n)$  with high probability. Let  $A$  be the size of the initial majority opinion. From (1) we obtain that opinion  $A$  wins with probability  $|A|/n$ . Thus if  $|A| = cn$ , opinion  $A$  wins with probability  $c < 1$ , even if  $A$  is a clear majority.
- $k = 2$ . This is the topic of this paper. We show that if the initial imbalance between the opinions is not too small, then with high probability the time to completion is  $\Theta(\log n)$ , resulting in an exponential speed up over the case  $k = 1$ , and the majority wins. More details are given in the next section.
- $k \geq 5$ . It follows from the proof presented by Abdullah and Draief [1], that for  $k$  odd and  $d \geq k$  constant, if the initial allocation of the opinions is chosen randomly, the initial imbalance is sufficiently large, and the selection of  $k$  neighbours is done without replacement, then with high probability the majority wins, and the voting completes in  $\Theta(\log \log n)$  rounds.

In the particular case of the complete graph  $K_n$ , the performance of two-party voting is well studied. Becchetti et al. [4] consider the case  $k = 3$  in these graphs. The main focus of the work is on the completion time as a function of the number of opinions. The result for two opinions is  $O(\log n)$  provided the difference is not too small. Cruise and Ganesh [11] consider a more general but asynchronous model. Their work includes the case  $k = 2$ , and gives a  $\Theta(\log n)$  result. A variant of two-sample voting has been considered by Doerr et al. [13], where the number of opinions can take any value from  $\{1, \dots, n\}$ . In their model, whenever a node  $v$  contacts two neighbors  $u$  and  $w$ , it adopts the median of the opinions of  $u$ ,  $v$  and  $w$ . Once the system is left with two opinions, this protocol is equivalent to the two-sample voting considered in this paper.

If initially there are  $s$  opinions, they showed an  $O(\log s \log \log n + \log n)$  convergence time to a so-called “stable consensus” on complete graphs.

An alternative approach to  $k$ -sample voting is to use a majority dynamic. In this case each vertex adopts the most popular opinion among *all* its neighbours. In [1, 22] the authors answered several important questions w.r.t. majority voting in general graphs, expanders, and random graphs, which we now describe.

Majority dynamics were studied by Mossel et al. in [22] who gave bounds for different scenarios. They consider a model where initially an opinion from  $\{1, \dots, k\}$  is assigned to the vertices independently according to a probability distribution. Then, the following deterministic process is considered, which is fully defined by the initial distribution. For  $T$  time steps, each vertex adopts the opinion held by the majority of its neighbors. After step  $T$  a fair and monotone election function is applied to the opinions of the vertices, resulting in a winning opinion. Mossel et al. [22] showed that (under certain assumptions) for the two-party model this process results in the correct (initial majority) answer.

Recently, majority voting was considered by Abdullah and Draief [1] on fixed degree sequence random graphs. They studied this process in the two-party case, where each vertex adopts the most popular opinion among the neighbors in each step. The initial opinions are distributed randomly according to a biased distribution. They showed that if the initial bias toward one opinion is large enough, then with high probability this opinion is adopted by all vertices within  $O(\log \log n)$  time steps, and established a similar lower bound.

## 2 Our results for two-sample voting

Assume that initially each vertex holds one of two opinions. For convenience,  $A$  and  $B$  will denote the opinions, the two sets of vertices who have these opinions, and the sizes of these sets, depending on the context. If opinion  $A$  is the majority, then the imbalance  $\nu$  (also referred to as the relative difference between the votes, or the advantage of the  $A$  vote) is given by  $A - B = \nu n$ .

As mentioned earlier, for single-sample pull voting on random  $d$ -regular graphs and expanders, the expected time to complete is  $\Theta(n)$  with high probability<sup>1</sup>. Moreover, for any connected regular graph, the probability that the initial majority  $A$  wins the vote is only  $A/n$ . In this paper we show that if there is a sufficient initial imbalance between the two opinions, then with high probability two-sample voting on  $d$ -regular random graphs and expanders is completed in a time which is logarithmic in the graph size, and the initial majority opinion wins.

Our results depend on the initial imbalance  $\nu_0$  and, in the case of random regular graphs on the degree  $d$  of the graph, while in the case of expanders on the second largest eigenvalue of the transition matrix. A random  $d$ -regular graph is a graph sampled uniformly at random from the set of all  $d$ -regular graphs. The results hold with high probability, which depends on the selection of a graph (in the case of random graphs) as well as on the voting process, which is itself probabilistic.

**Theorem 1** *Let  $G$  be a random  $n$ -vertex  $d$ -regular graph with opinions  $A$  and  $B$  and with initial imbalance  $\nu_0 = |A - B|/n$ . There is an absolute constant  $K$  (independent of  $d$ ) such that, provided*

$$\nu_0 \geq K \sqrt{\frac{d}{n} + \frac{1}{d}}, \quad (2)$$

*with high probability two-sample voting is completed in  $O(\log n)$  steps, and the winner is the opinion with the initial majority.*

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<sup>1</sup>With high probability or w.h.p. means with probability tending to 1 as  $n$  increases.

**Corollary 1** (*Sparse random graphs.*) *Let  $G$  be a random  $d$ -regular graph with  $d \leq \sqrt{n}$ , and let opinions  $A$  and  $B$  be placed on the vertices of  $G$ . There is a constant  $K$  such that, provided*

$$\nu_0 \geq \frac{K}{\sqrt{d}}, \quad (3)$$

*with high probability two-sample voting is completed in  $O(\log n)$  steps and the winner is the opinion with the initial majority.*

We give a similar result for expanders, that is, for a  $d$ -regular graph  $G$  with a small second eigenvalue  $\lambda_G = \max\{\lambda_2, |\lambda_n|\}$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  are the eigenvalues of the transition matrix  $P = (1/d)A$  of a random walk on  $G$  and  $A$  is the adjacency matrix of  $G$ .

**Theorem 2** *Let  $G$  be an  $n$ -vertex  $d$ -regular graph and let opinions  $A$  and  $B$  be placed on the vertices of  $G$ . There is an absolute constant  $K$  (independent of  $d$  and  $\lambda_G$ ) such that, provided*

$$\nu_0 \geq K\lambda_G,$$

*with high probability two-sample voting is completed in  $O(\log n)$  steps and the winner is the opinion with the initial majority.*

Observe that for the above results to be non-trivial, we should consider  $K^2 \leq d \leq n/K^2$  in Theorem 1,  $K^2 \leq d \leq \sqrt{n}$  in Corollary 1, and  $\lambda_G \leq 1/K$  in Theorem 2. According to the results above, in order to guarantee that two-sample voting (starting with small imbalance  $\nu_0$ ) completes within  $O(\log n)$  steps and the initial majority opinion wins, it suffices to take a random  $d$ -regular graph with appropriately large degree  $d$ , or an expander graph appropriately small  $\lambda_G$ . We also show that the initial majority wins in  $O(\log n)$  steps in random regular graphs with small degree as well as in expanders with  $\lambda_G$  not too small, if the initial minority is a small constant fraction of the number of vertices.

**Theorem 3** *Let  $d > 10$  and let  $G$  be a random  $d$ -regular  $n$ -vertex graph with votes  $A$  and  $B$ . There is a constant  $c > 0$  (independent of  $d$ ) such that, provided the initial size of the minority vote  $B$  is at most  $cn$ , with high probability two-sample voting is completed in  $O(\log n)$  steps and the winner is the initial majority opinion  $A$ .*

**Theorem 4** *Let  $G$  be a  $d$ -regular  $n$ -vertex graph with  $\lambda = \lambda_G = 3/5 - \epsilon$ . Then, provided the initial size of the minority vote  $B$  is at most  $(\epsilon/5)n$ , with high probability two-sample voting is completed in  $O(\log n)$  steps and the winner is the initial majority opinion  $A$ .*

The above theorems hold under following adversarial conditions. The adversary has full knowledge of the graph, decides the initial distribution of the opinions among the vertices, and can arbitrarily redistribute the opinions at the beginning of each voting step. The adversary cannot change the number of opinions of each type. However, one can trace our proofs to see that if we allow the adversary to change the opinions of at most  $f = o(\nu_0 n)$  vertices during the execution of the algorithm, then under the conditions considered in Theorems 1 and 2 and Corollary 1, after  $O(\log n)$  voting steps all but  $O(f)$  vertices adopt the majority opinion  $A$ . That is, with high probability the protocol can suppress persistently corrupted vertices (cf. [13]).

As described in the previous section, it seems that there is a clearly defined hierarchy w.r.t. distributed voting in random regular graphs. If every node is only allowed to consult one single neighbour (and adopt its opinion), then – as shown in [8] – one requires  $\Theta(n)$  steps

on the graphs we consider to converge to one opinion. If every node can consult two neighbours (selecting them randomly with or without replacement) and adopt the opinion of these two vertices if they are the same, then the running time is  $O(\log n)$ , so exponentially faster. On the other side, even if the adversary is not allowed to re-distribute votes,  $\Omega(\log_d n)$  is a natural lower bound in any  $d$ -regular graph. This holds since there might be initially  $\Theta(n)$  pairs of adjacent vertices all with the same minority opinion  $B$ . The vertices of such a pair choose each other with probability  $\Theta(1/d^2)$ , in which case none of them will change its opinion. Thus, the protocol needs  $\Omega(\log_d n)$  steps in order to guarantee that in none of these  $\Theta(n)$  pairs the vertices choose each other all the time. This lower bound holds also for  $k$ -sample voting for a constant  $k \geq 3$ , if the selection of  $k$  neighbours is done with replacement (if a  $B$  vertex  $v$  has a  $B$  neighbour, then  $v$  does not change its opinion in the current step with probability  $\Omega(1/d^k)$ ). If every node may contact at least five different neighbours (selection without replacement) and adopt the majority opinion among them, then on random regular graphs with randomly distributed opinions (biased toward  $A$ ),  $\Theta(\log \log n)$  steps suffice until  $A$  wins (this follows from the analysis in [1]).

We should mention that the last result does not hold if the opinions are not randomly distributed. An adversary could assign the minority opinion  $B$  to a vertex  $v$  as well as all vertices which are at distance at most  $\text{Diam}/3$  to  $v$ , where  $\text{Diam}$  denotes the diameter of the graph. Clearly, the voting protocol needs at least  $\text{Diam}/3$  steps. Also, the example in the next paragraph shows that the result w.r.t. 5-sample voting [1] cannot be extended to graphs with similar conductance as in random graphs.

Consider a random regular graph of degree  $d - 5$  with  $n$  vertices, where  $d = \omega(1)$ . Then, we group the vertices in clusters of size 6 – we assume that  $n$  is a multiple of 6. In each cluster, we connect every vertex with all the other 5 vertices, so the graph has degree  $d$ . Bollobás [5] shows that the conductance of this graph is with high probability at least  $1/2 - o(1)$  and at most  $1/2(1 + o(1)) + 5/d$ . The upper bound holds since for any subset  $S$  with  $|S| \leq n/2$ , there are at least  $(1 - o(1))(d - 5)|S|/2$  edges crossing the cut between  $S$  and  $V \setminus S$  in a  $d - 5$ -regular random graph [5]. On the other hand, there is a set for which the size of the cut is at most  $(1 + o(1))(d - 5)|S|/2$ . Furthermore, each node has 5 additional (inner-cluster) edges, which may increase the cut. Since  $d = \omega(1)$ , the conductance of this graph is with high probability  $1/2 \pm o(1)$ , which is almost the same as in a  $d$ -regular random graph. Clearly, all vertices from a cluster may choose each other in a step with some probability larger than  $1/d^{30}$ . Thus, the protocol needs  $\Omega(\log_d n)$  steps in order to guarantee that in none of these clusters the vertices choose each other all the time.

Concerning our results, the constant eigenvalue gap  $1 - \lambda_G$  (as in Theorem 4) seems to be needed. For example, consider a hypercube with  $d = \log n$  and  $1 - \lambda_G = o(1)$ . If the adversary is allowed to rearrange the opinions in each step, then we may have for  $\Omega(d^2)$  steps configurations, in which all vertices of a subcube of dimension  $d - c$  have opinion  $B$ , where  $c$  is a constant. Such a  $B$ -vertex converts to  $A$  with probability  $(c/d)^2$ , so  $\Omega(d^2) = \Omega(\log^2 n)$  steps are needed for the protocol to finish.

The proof techniques used for single-sample voting, namely coalescing random walks, do not apply in the case of two-sample voting. Our proofs are based on concentration of the size of edge cuts around the expectation coupled with a worst case analysis.

### 3 Background material and outline of proof

The analysis of the voting process is made in the following three phases, where  $B$  is the minority vote.



**Phase I:**  $cn \leq B \leq n(1 - \nu_0)/2$ .

**Phase II:**  $\omega \leq B \leq cn$ .

**Phase III:**  $1 \leq B \leq \omega$ .

Let  $B(t)$  denote the set of vertices with opinion  $B$  and the size of this set in step  $t$ . Whenever it is clear from the context, we write  $B$  instead of  $B(t)$ . Phase I reduces  $B(t)$  from  $B(0) = n(1 - \nu_0)/2$  to  $B(T) \leq cn$ , for some small constant  $c$ , in a sequence of  $T = O(\log(1/\nu_0))$  rounds. The reduction of  $B(t)$  in Phase II is more dramatic. The  $\omega$  threshold between phases II and III is a function slowly growing with  $n$ . In Phase III things may slow down again and the last few steps can be viewed as a biased random walk. All phases are analysed in the adversarial model, which allows an arbitrary redistribution of the votes at the beginning of each step.

The following Chernoff–Hoeffding inequalities are used throughout the proofs. Let  $Z = Z_1 + Z_2 + \dots + Z_N$  be the sum of the independent random variables  $0 \leq Z_i \leq 1$ ,  $i = 1, 2, \dots, N$ ,  $\mathbf{E}(Z_1 + Z_2 + \dots + Z_N) = N\mu$ , and  $0 \leq \epsilon \leq 1$ . Then

$$\Pr(Z \leq (1 - \epsilon)N\mu) \leq e^{-\epsilon^2 N\mu/3}, \quad (4)$$

$$\Pr(Z \geq (1 + \epsilon)N\mu) \leq e^{-\epsilon^2 N\mu/2}. \quad (5)$$

For any  $\epsilon > 0$ , we have

$$\Pr(Z \geq (1 + \epsilon)N\mu) \leq \left( \frac{e^\epsilon}{(1 + \epsilon)^{1+\epsilon}} \right)^{N\mu}. \quad (6)$$

Our proofs for the case of random graphs are made using the configuration model of  $d$ -regular  $n$ -vertex multigraphs. Let  $\mathcal{C}_{n,d}$  be the space of  $d$ -regular  $n$ -vertex configurations, and let  $\mathcal{C}_{n,d}^*$  be the sub-space of  $\mathcal{C}_{n,d}$  of the configurations whose underlying graphs are simple. A configuration  $S$  is a matching of the  $nd$  “configuration points” (each vertex is represented by  $d$  points). Every simple graph maps to the same number of configurations, so  $\mathcal{C}_{n,d}^*$  maps uniformly onto  $\mathcal{G}_{n,d}$ , the space of  $d$ -regular  $n$ -vertex graphs. We use the following result of [16] for the size of  $|\mathcal{C}^*|/|\mathcal{C}|$ . See e.g. [10] for a proof.

**Lemma 1** *Let  $1 \leq d \leq n/8$ . If  $S$  is chosen uniformly at random from  $\mathcal{C}_{n,d}$ , then*

$$\Pr(S \in \mathcal{C}_{n,d}^*) \geq e^{-20d^2}. \quad (7)$$

This lemma is used in the following way. Let  $\mathbf{Q}$  be a property of  $d$ -regular  $n$  vertex multigraphs. Then, denoting by  $G(S)$  the underlying multigraph of configuration  $S$ ,

$$\begin{aligned} \Pr_{\mathcal{G}}(G \in \mathbf{Q}) &= \Pr_{\mathcal{C}}(G(S) \in \mathbf{Q} \mid S \in \mathcal{C}^*) \leq \frac{\Pr_{\mathcal{C}}(G(S) \in \mathbf{Q})}{\Pr_{\mathcal{C}}(S \in \mathcal{C}^*)} \\ &\leq \Pr_{\mathcal{C}}(G(S) \in \mathbf{Q}) \cdot e^{20d^2}. \end{aligned} \quad (8)$$

At any step  $t$  of the voting process, let  $\Delta_{AB} = \Delta_{AB}(t)$  be the number of  $A$  vertices converting to  $B$  during this step. Similarly, let  $\Delta_{BA}$  be the number of  $B$  vertices converting to  $A$  during step  $t$ . At each step we obtain a lower bound on  $\mathbf{E}\Delta_{BA}$ , an upper bound on  $\mathbf{E}\Delta_{AB}$ , and use the concentration of these two random variables given by (4) and (5) to get a w.h.p. value of  $\Delta = \Delta_{BA} - \Delta_{AB}$ , which is the increase of the number of  $A$  vertices in this step.

For a vertex  $v$  and a set of vertices  $C$ , let  $d_v^C$  be the number of vertices in  $C$  which are adjacent to  $v$ . For  $v \in A$ , let  $X_v = 1$  if  $v$  chooses twice in  $B$  at step  $t$ , and 0 otherwise. Thus

$$\Delta_{AB} = X_A = \sum_{v \in A} X_v$$

The  $X_v$  are independent  $\{0, 1\}$  random variables with the expected value depending whether the neighbours are selected with or without replacement:

$$\mathbf{E}X_v(\text{with replacement}) = \left(\frac{d_v^B}{d}\right)^2, \quad \mathbf{E}X_v(\text{no replacement}) = \frac{(d_v^B)(d_v^B - 1)}{d(d - 1)}.$$

We give proofs for sampling with replacement. The proofs for sampling without replacement follow because

$$\mathbf{E}X_v(\text{no replacement}) = \frac{d}{d - 1} \mathbf{E}X_v(\text{with replacement}),$$

so all inequalities for expected values in one model imply similar inequalities in the other model.

## 4 Phase I of analysis: $cn \leq B \leq n(1 - \nu_0)/2$

### 4.1 The main lemma and its applications to expanders

Lemma 2 below gives a sufficient condition for a fast reduction of the minority  $B$ -vote from  $(1 - \nu_0)n/2$  to  $cn$ , where  $\nu_0 < 1$  and  $c < (1 - \nu_0)/2$ . For example, for  $\nu_0 = 1/10$  and  $c = 1/20$ , the  $B$  vote reduces from  $(9/20)n$  to  $(1/20)n$ . The condition in Lemma 2 says that the number  $E(X, Y)$  of edges between any disjoint large subsets of vertices  $X$  and  $Y$  is close to the value  $dXY/n$  expected in the random regular graph. This condition is of the form as in the Expander Mixing Lemma (stated below as Lemma 3), so Lemma 2 can be immediately applied to expanders (see Corollary 2). Lemma 2 can also be applied without a reference to the second eigenvalue (if the second eigenvalue is not known or is not good enough) by directly checking that large subsets of vertices are connected by many edges. We illustrate this by considering random  $d$ -regular graphs for any  $d \in [K, n/K]$ , where  $K$  is some (large) constant (see Lemma 4 and Corollary 3).

**Lemma 2** *Let  $0 < c \leq 1/2$ ,  $0 < \alpha \leq c^{3/2}/36$ , and  $\alpha^2 c^2 n = \Omega(n^\epsilon)$ , for a constant  $\epsilon > 0$ . Let  $G$  be a  $d$ -regular  $n$ -vertex connected graph such that*

$$\left| E(X, Y) - \frac{dXY}{n} \right| \leq \alpha d \sqrt{XY}, \quad (9)$$

*for each pair  $X$  and  $Y$  of disjoint subsets of vertices of sizes  $Y \geq cn$  and  $X \geq (2/3)\alpha c^{3/2}n$ . There exist absolute constants  $K$  and  $K'$  (independent of  $d$ ,  $c$  and  $\alpha$ ) such that, if the initial advantage of the  $A$ -vote in  $G$  is*

$$\nu_0 \geq K\alpha, \quad (10)$$

*then with probability at least  $1 - e^{-\Theta(\alpha^2 c^2 n)}$ , the advantage of the  $A$ -vote increases to  $1 - 2c$  (that is, the  $B$ -vote decreases to  $cn$ ) within  $K'(\log(1/\nu_0) + \log(1/c))$  voting steps.*

The parameters  $c$  and  $\alpha$  in the above lemma can be considered as some small constants, but they can also depend on  $d$  (and decrease with increasing  $d$ ).

**Lemma 3** (Expander Mixing Lemma [3]). *Let  $G = (V, E)$  be a  $d$ -regular  $n$ -vertex graph. Let  $1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq -1$  be the eigenvalues of the transition matrix of the random walk on  $G$ , and let  $\lambda = \lambda_G = \max\{|\lambda_2|, |\lambda_n|\}$ . Then for all  $S, T \subseteq V$ ,*

$$\left| E(S, T) - \frac{dST}{n} \right| \leq \lambda d \sqrt{ST}.$$



Lemmas 2 and 3 imply the following corollary.

**Corollary 2** *For any constant  $0 < c < 1/2$ , there exist constants  $K_1$  and  $K_2$  (which depend on  $c$ ) such that for any regular  $n$ -vertex graph  $G$  with the initial advantage of the  $A$ -vote  $\nu_0 \geq K_1\lambda$ , the minority vote  $B$  decreases to  $cn$  within  $K_2 \log(1/\nu_0)$  voting steps, with probability at least  $1 - e^{-\Theta(\lambda^2 n)}$ .*

*Proof.* Let  $K_1 = \max\{K, 36/c^{3/2}\}$ , where constant  $K$  is from Lemma 2. If  $\lambda \geq 1/K_1$ , then the statement of the corollary is trivially fulfilled. If  $\lambda \leq 1/K_1$ , then  $\lambda \leq c^{3/2}/36$  and we can apply Lemma 2 with  $c$  and  $\alpha = \lambda$ . Now Lemmas 2 and 3 imply that if  $\nu_0 \geq K_1\lambda$ , then with probability at least  $1 - e^{-\Theta(\lambda^2 n)}$ , the size of the  $B$ -vote decreases to  $cn$  within  $K'(\log(1/\nu_0) + \log(1/c)) = K_2(\log(1/\nu_0))$  voting steps.  $\square$

## 4.2 Application to random regular graphs

If  $d = O(1)$ , then a random  $d$ -regular graph has  $\lambda_G \leq (2\sqrt{d-1} + \epsilon)/d$ , w.h.p., where  $\epsilon > 0$  can be any small constant [17]. Thus, for  $d = O(1)$  Corollary 2 applies. To apply Lemma 2 to random regular graphs with degree which may grow with the number of vertices, we need to establish a suitable  $\alpha$  for the bound in (9) without referring to  $\lambda_G$ . The bound we show in the next lemma is stronger than a similar bound shown by Fountoulakis and Panagiotou [15]. Using the bound from [15] would lead to a weaker relation between  $\nu_0$  and  $d$  than in Theorem 1.

**Lemma 4** *For given set sizes  $X \leq Y$ , in a random  $d$ -regular  $n$ -vertex graph  $G = (E, V)$ , with probability at least  $1 - 2e^{-Y}$ , each pair of disjoint subsets of vertices  $\mathcal{X}$  and  $\mathcal{Y}$  of sizes  $X$  and  $Y$  satisfies the following inequality:*

$$\left| E(\mathcal{X}, \mathcal{Y}) - \frac{dXY}{n} \right| \leq d\sqrt{XY} \sqrt{\frac{1}{d} 24 \log(ne/Y) + \frac{d}{Y} 160}. \quad (11)$$

**Corollary 3** *For any constant  $0 < c < 1/2$ , there exist constants  $K_1$  and  $K_2$  (which depend on  $c$ ) such that for a random  $d$ -regular  $n$ -vertex graph with the initial advantage of the  $A$ -vote*

$$\nu_0 \geq K_1 \sqrt{\frac{1}{d} + \frac{d}{n}}, \quad (12)$$

*the minority vote  $B$  decreases within  $K_2 \log(1/\nu_0)$  steps to  $cn$ , with probability at least  $1 - e^{-\Theta(n^{1/2})}$ .*

*Proof.* Let  $0 < c < 1/2$  be a constant and let

$$\alpha = \sqrt{\frac{1}{d} 24 \log(e/c) + \frac{d}{n} (160/c)} = \Theta \left( \sqrt{\frac{1}{d} + \frac{d}{n}} \right). \quad (13)$$

Lemma 4 implies that for a random  $d$ -regular  $n$ -vertex graph, the probability that Inequality (9) holds for each pair  $X$  and  $Y$  of disjoint subsets of vertices such that  $Y \geq cn$  is at least  $1 - 2n^2 e^{-cn} = 1 - e^{-\Theta(n)}$ . If Inequality (9) holds for all such pairs of subsets of vertices, then Lemma 2 and (13) imply that there are constants  $K_1$  and  $K_2$  such that if the initial vote imbalance is  $\nu_0 \geq K\alpha \geq K_1 \sqrt{1/d + d/n}$ , then with probability at least  $1 - e^{-\Theta(\alpha^2 n)} \geq 1 - e^{-\Theta(n^{1/2})}$ , the minority vote  $B$  decreases to  $cn$  in  $K_2 \log(1/\nu_0)$  steps. Thus with probability at least  $(1 - e^{-\Theta(n)})(1 - e^{-\Theta(n^{1/2})}) = 1 - e^{-\Theta(n^{1/2})}$ , for a random  $d$ -regular  $n$ -vertex graph with the initial advantage of the  $A$ -vote  $\nu_0 \geq K_1 \sqrt{1/d + d/n}$ , the minority vote reduces to  $cn$  in  $K_2 \log(1/\nu_0)$  steps.  $\square$

### Proof of Lemma 4

Let  $X, Y$  be two fixed disjoint vertex sets with sizes  $X \leq Y$  ( $X$  and  $Y$  stand for the sets and their sizes). Let  $Z(S)$  be the number of edges between  $X$  and  $Y$  in a configuration  $S \in \mathcal{C}_{n,d}$ . We order the  $nd$  configuration points so that the first  $dX$  points correspond to the vertices in set  $X$ . A configuration  $S$  can be represented as a sequence  $(t_1, t_2, \dots, t_q)$ , where  $q = nd/2$ ,  $1 \leq t_i \leq q - (2i - 1)$ , and the number  $t_i$  defines the  $i$ -th (matched) pair in  $S$ , assuming the lexicographic order of pairs. Denoting by  $L_i$  the sequence of the remaining unmatched points after the first  $(i - 1)$  pairs have been selected, the  $i$ -th pair matches the first point in  $L_i$  (which becomes the first point of this pair) with the point in  $L_i$  at the position  $1 + t_i$ . A random  $S \in \mathcal{C}_{n,d}$  is determined by independent random selections of  $t_i$ 's. By considering first the points corresponding to the vertices in  $X$ , we ensure that  $Z(S)$  is determined by  $(t_1, t_2, \dots, t_{dX})$ .

For a configuration  $S = (t_1, t_2, \dots, t_q)$  and  $i = 0, 1, \dots, q$ , let

$$Z_i(S) \equiv Z_i(t_1, t_2, \dots, t_q) = \mathbf{E}_{\tau_{i+1}, \dots, \tau_q} Z(t_1, \dots, t_i, \tau_{i+1}, \dots, \tau_q) \equiv Z(t_1, t_2, \dots, t_i).$$

That is,  $Z_i(S)$  is the expected number of edges between the  $X$  and  $Y$  points in a random configuration which agrees with the configuration  $S$  on the first  $i$  pairs.

We have  $Z_0(S) = \mathbf{E}Z(S) = dXY/n$  and  $Z_{dX}(S) = Z(S)$ . The sequence of random variables  $Z_i$ ,  $i = 0, 1, \dots, q$  is a martingale because  $\mathbf{E}(Z_{i+1}|Z_i) = Z_i$ :

$$\begin{aligned} \mathbf{E}(Z_{i+1}|Z_i = z) &= \mathbf{E}(Z(t_1, \dots, t_{i+1})|Z(t_1, \dots, t_i) = z) \\ &= \sum_{t_1, \dots, t_{i+1}: Z(t_1, \dots, t_i) = z} \frac{\mathbf{Pr}(t_1, \dots, t_{i+1})}{\mathbf{Pr}(Z_i = z)} Z(t_1, \dots, t_{i+1}) \\ &= \frac{1}{\mathbf{Pr}(Z_i = z)} \sum_{t_1, \dots, t_i: Z(t_1, \dots, t_i) = z} \mathbf{Pr}(t_1, \dots, t_i) \sum_{t_{i+1}} \mathbf{Pr}(t_{i+1}) Z(t_1, \dots, t_{i+1}) \\ &= \frac{1}{\mathbf{Pr}(Z_i = z)} \sum_{t_1, \dots, t_i: Z(t_1, \dots, t_i) = z} \mathbf{Pr}(t_1, \dots, t_i) Z(t_1, \dots, t_i) \\ &= z. \end{aligned}$$

Let  $F_i^X(S)$  and  $F_i^Y(S)$  denote the number of available (unmatched)  $X$ -points and  $Y$ -points, respectively, after the first  $i$  pairs in  $S$  have been matched. If  $F_i^X(S) = 0$ , then  $Z_j(S) = Z(S)$ , for all  $j \geq i$ . If  $F_i^X(S) \geq 1$ , then, dropping  $S$  from the notation for simplicity,

$$Z_i = (Y - F_i^Y) + \frac{F_i^X F_i^Y}{nd - (2i + 1)}.$$

The first term  $Y - F_i^Y$  is the number of edges between the  $X$  and  $Y$  points given by the first  $i$  pairs in  $S$ , and the second term is the expected number of edges between the  $X$  and  $Y$  points contributed by a random matching of the remaining points. (Each available point  $x$  in  $X$  contributes  $F_i^Y/(nd - (2i + 1))$  to this expectation, because this is the probability that  $x$  is matched with a point in  $Y$ .) When the next  $(i + 1)$ -st pair is matched, then  $(F_{i+1}^X, F_{i+1}^Y)$  is either  $(F_i^X - 1, F_i^Y - 1)$ , or  $(F_i^X - 2, F_i^Y)$ , or  $(F_i^X - 1, F_i^Y)$ , and it can be checked that in all three cases

$$|Z_{i+1} - Z_i| \leq 2. \tag{14}$$

Alternatively, the bound (14) can be established by applying the general *switching method* (see [25] for discussion of this method). If two configurations  $S'$  and  $S''$  differ only by two pairs,

that is,  $S''$  can be obtained from  $S'$  by “switching” two pairs  $(a, b)$  and  $(c, d)$ , where  $a < c$ , to pairs  $(a, c)$  and  $\{b, d\}$ , or  $(a, d)$  and  $\{b, c\}$ , then clearly

$$|Z(S') - Z(S'')| \leq 2. \quad (15)$$

(Note that we use the set notation for the pairs  $\{b, d\}$  and  $\{b, c\}$ , because we do not know the relative order of points  $b$  and  $d$ , and  $b$  and  $c$ .)

Define  $\mathcal{C}_i(S)$  as the set of all configurations  $S'$  which have the same first  $i$  pairs as in  $S$ . For a configuration  $S' = ((a'_1, a'_2), \dots, (a'_{nd-1}, a'_{nd}))$ ,  $1 \leq i < nd/2$  and  $2i \leq j \leq nd$ , let  $S'[i, j]$  be the configuration obtained from  $S'$  by switching the  $i$ -th pair  $(a'_{2i-1}, a'_{2i})$  and the pair  $\{a'_j, b\}$  for the pairs  $(a'_{2i-1}, a'_j)$  and  $\{a'_{2i}, b\}$  (note that  $S'[i, 2i] = S'$ ). The set  $\mathcal{C}_i(S)$  can be obtained from the set  $\mathcal{C}_{i+1}(S)$  by replacing each configuration  $S' \in \mathcal{C}_{i+1}(S)$  with the configurations  $S'[i+1, j]$ , for  $j = 2i+2, \dots, nd$ :

$$\mathcal{C}_i(S) = \{S'[i+1, j] : S' \in \mathcal{C}_{i+1}(S), 2i+2 \leq j \leq nd\}. \quad (16)$$

Using (16), we can write  $Z_{i+1}(S) - Z_i(S)$  as

$$\begin{aligned} Z_{i+1}(S) - Z_i(S) &= \frac{1}{|\mathcal{C}_{i+1}(S)|} \sum_{S' \in \mathcal{C}_{i+1}(S)} Z(S') - \frac{1}{|\mathcal{C}_i(S)|} \sum_{S' \in \mathcal{C}_i(S)} Z(S') \\ &= \frac{nd - (2i+1)}{|\mathcal{C}_i(S)|} \sum_{S' \in \mathcal{C}_{i+1}(S)} Z(S') - \frac{1}{|\mathcal{C}_i(S)|} \sum_{S' \in \mathcal{C}_{i+1}(S)} \sum_{j=2i+2}^{nd} Z(S'[i+1, j]) \\ &= \frac{1}{|\mathcal{C}_i(S)|} \sum_{j=2i+2}^{nd} \sum_{S' \in \mathcal{C}_{i+1}(S)} (Z(S') - Z(S'[i+1, j])). \end{aligned}$$

There are  $|\mathcal{C}_i(S)|$  terms in the last double sum and the absolute value of each term is at most 2 (Inequality (15)), so the bound (14) follows.

The Azuma-Hoeffding inequality says that if a sequence of random variables  $(X_0, X_1, \dots, X_N)$  is a martingale and  $|X_{i+1} - X_i| \leq c$ , for each  $1 \leq i \leq N-1$ , then for any  $\delta$ ,

$$\Pr(|X_N - X_0| \geq \delta) \leq 2 \exp\left(-\frac{\delta^2}{2Nc^2}\right).$$

Applying this inequality to our martingale  $(Z_0, Z_1, \dots)$ , we get

$$\Pr_{\mathcal{C}}\left(\left|E(X, Y) - \frac{dXY}{n}\right| \geq \delta\right) = \Pr(|Z_{dX}(S) - Z_0(S)| \geq \delta) \leq 2 \exp\left(-\frac{\delta^2}{8dX}\right).$$

Thus for a random  $d$ -regular  $n$ -vertex graph  $G = (E, V)$ , using (8),

$$\Pr_{\mathcal{G}}\left(\left|E(X, Y) - \frac{dXY}{n}\right| \geq \delta\right) \leq 2 \exp\left(-\frac{\delta^2}{8dX} + 20d^2\right).$$

The number of pairs of disjoint sets of sizes  $X \leq Y$  is at most

$$\binom{n}{X} \binom{n-X}{Y} \leq \left(\frac{ne}{X}\right)^X \left(\frac{ne}{Y}\right)^Y \leq \left(\frac{ne}{Y}\right)^{2Y}.$$

The last inequality holds because  $(ne/z)^z$  is monotone increasing for  $0 \leq z \leq n$ . Therefore, using the union bound, for given set sizes  $X \leq Y$  and a random  $d$ -regular  $n$ -vertex graph  $G = (E, V)$ ,

$$\begin{aligned} \mathbf{Pr}(\text{there are disjoint } \mathcal{X}, \mathcal{Y} \subseteq V \text{ of sizes } X \text{ and } Y \text{ such that } |E(\mathcal{X}, \mathcal{Y}) - dXY/n| \geq \delta) \\ \leq 2 \exp \left( -\frac{\delta^2}{8dX} + 20d^2 + 2Y \log(ne/Y) \right). \end{aligned}$$

The above bound is at most  $2e^{-Y}$ , if

$$\frac{\delta^2}{8dX} - 20d^2 - 2Y \log(ne/Y) \geq Y,$$

which is equivalent to

$$\delta \geq d\sqrt{XY} \sqrt{\frac{1}{d}(16 \log(ne/Y) + 8) + \frac{d}{Y}160}. \quad (17)$$

Since the right-hand side in (11) is at least the right-hand side in (17), we conclude that (11) holds for all pairs of disjoint sets  $\mathcal{X}$  and  $\mathcal{Y}$  of sizes  $X \leq Y$  with probability at least  $1 - 2e^{-Y}$ .

### 4.3 Proof of Lemma 2

Let  $c$  and  $\alpha$  be as in the statement of the lemma. Let  $A$  and  $B$  be the voting groups at the beginning of the current step, let  $\nu = (A - B)/n$ , and assume that for some sufficiently large constant  $K$ ,

$$K\alpha \leq \nu \leq 1 - 2c, \quad (18)$$

or equivalently,

$$cn \leq B \leq \frac{1 - K\alpha}{2}n.$$

**Lower bound on  $\Delta_{BA}$ .** The expectation  $\mathbf{E}\Delta_{BA}$ , taken over the random selection of neighbours in the current voting step, is equal to

$$\begin{aligned} \mathbf{E}\Delta_{BA} &= \sum_{v \in B} \left( \frac{d_v^A}{d} \right)^2 \geq \frac{1}{Bd^2} \left( \sum_{v \in B} d_v^A \right)^2 = \frac{1}{Bd^2} (E(A, B))^2 \\ &\geq \frac{1}{Bd^2} \left( \frac{dAB}{n} - \alpha d\sqrt{AB} \right)^2 = \frac{A^2B}{n^2} \left( 1 - \alpha \frac{n}{\sqrt{AB}} \right)^2 \end{aligned} \quad (19)$$

$$\geq \frac{A^2B}{n^2} \left( 1 - 2\alpha \frac{n}{\sqrt{AB}} \right) = \frac{A^2B}{n^2} (1 - 2\eta), \quad (20)$$

where  $\eta \equiv \eta_{AB} = \alpha n / \sqrt{AB} \leq \alpha \sqrt{2/c} \leq c/25$ . The inequality on line (19) is the inequality (9) applied to sets  $A$  and  $B$  (as  $X$  and  $Y$ ). Therefore,

$$\mathbf{Pr} \left( \Delta_{BA} \leq \frac{A^2B}{n^2} (1 - 3\eta) \right) \leq \mathbf{Pr} (\Delta_{BA} \leq (1 - \eta) \mathbf{E}\Delta_{BA}) \leq e^{-\eta^2 cn/24} \leq e^{-\alpha^2 cn/6}. \quad (21)$$

The second inequality above follows from (4) applied with  $\epsilon = \eta$ , and the fact that  $\mathbf{E}\Delta_{BA} \geq (1/8)cn$  (from (20)). The last inequality in (21) holds because the definition of  $\eta$  implies

$$2\alpha \leq \eta \leq \frac{\alpha}{\sqrt{(1-c)c}}. \quad (22)$$

**Upper bound on  $\Delta_{AB}$ .** Let  $q = \lfloor (1/2) \log_2(n^2/(\eta B^2)) \rfloor + 1$ . We partition set  $A$  into  $q + 1$  sets  $C_i$ , according to the values  $d_v^B$ :

$$\begin{aligned} C_0 &= \left\{ v \in A : d_v^B < (1 + \eta) \frac{dB}{n} \right\}; \\ C_i &= \left\{ v \in A : (1 + 2^{i-1}\eta) \frac{dB}{n} \leq d_v^B < (1 + 2^i\eta) \frac{dB}{n} \right\}, \quad \text{for } i = 1, 2, \dots, q-1; \\ C_q &= \left\{ v \in A : (1 + 2^{q-1}\eta) \frac{dB}{n} \leq d_v^B \right\}. \end{aligned}$$

Obviously  $C_0 \leq A$ , and we show that  $C_i \leq A/2^{2(i-1)}$ , for all  $1 \leq i \leq q$ . For each  $1 \leq i \leq q$ , we have

$$E(C_i, B) = \sum_{v \in C_i} d_v^B \geq (1 + 2^{i-1}\eta) \frac{dC_i B}{n}. \quad (23)$$

If we had  $C_i > A/2^{2(i-1)}$  for some  $1 \leq i \leq q$ , then, from the definition of  $q$  and  $\eta$ ,

$$C_i > A/2^{2(q-1)} \geq \eta AB^2/n^2 = \alpha A^{1/2} B^{3/2}/n \geq \alpha(2/3)c^{3/2}n.$$

Then applying (9) to sets  $C_i$  and  $B$  (as  $X$  and  $Y$ ) would give

$$\begin{aligned} E(C_i, B) &\leq \frac{dC_i B}{n} + \alpha d \sqrt{C_i B} = \left(1 + \alpha \frac{n}{\sqrt{C_i B}}\right) \frac{dC_i B}{n} \\ &< \left(1 + \alpha 2^{i-1} \frac{n}{\sqrt{AB}}\right) \frac{dC_i B}{n} = (1 + 2^{i-1}\eta) \frac{dC_i B}{n}. \end{aligned}$$

This would contradict (23).

The expectation  $\mathbf{E}\Delta_{AB}$ , taken over the random selection of neighbours in the current voting step, is equal to

$$\begin{aligned} \mathbf{E}\Delta_{AB} &= \sum_{v \in A} \left(\frac{d_v^B}{d}\right)^2 = \sum_{i=0}^q \sum_{v \in C_i} \left(\frac{d_v^B}{d}\right)^2 \leq C_q + \sum_{i=0}^{q-1} \sum_{v \in C_i} \left(\frac{d_v^B}{d}\right)^2 \\ &\leq C_q + \sum_{i=0}^{q-1} C_i \left(\frac{(dB/n)(1 + 2^i\eta)}{d}\right)^2 = C_q + \frac{B^2}{n^2} \sum_{i=0}^{q-1} C_i (1 + 2^{i+1}\eta + 2^{2i}\eta^2) \quad (24) \end{aligned}$$

$$\begin{aligned} &= C_q + \frac{B^2}{n^2} \left( \sum_{i=0}^{q-1} C_i + C_0(2\eta + \eta^2) + \eta \sum_{i=1}^{q-1} 2^{i+1} C_i (1 + 2^{i-1}\eta) \right) \\ &\leq C_q + \frac{B^2}{n^2} \left( A + A(2\eta + \eta^2) + \eta \sum_{i=1}^{q-1} \frac{A}{2^{i-3}} (1 + 2^{i-1}\eta) \right) \quad (25) \end{aligned}$$

$$\begin{aligned} &\leq C_q + \frac{AB^2}{n^2} (1 + 2\eta + \eta^2 + 8\eta + 4q\eta^2) \\ &\leq C_q + \frac{AB^2}{n^2} (1 + 11\eta) \quad (26) \end{aligned}$$

$$\leq \frac{AB^2}{n^2} (1 + 15\eta). \quad (27)$$

Inequality (24) follows from the definition of sets  $C_i$ . Inequality (25) holds since  $\sum_{i=0}^q C_i = A$ ,  $C_0 \leq A$ , and  $C_i \leq A/2^{2(i-1)}$ . To see that Inequality (26) holds, check that  $\eta(1+4q) \leq 1$ , using the definitions of  $\eta$  and  $q$ ,  $B \geq cn$  and the bound on  $\alpha$ . Finally, Inequality (27) holds since  $C_q \leq A/2^{2(q-1)} \leq 4\eta AB^2/n^2$ .

The above bound on  $\mathbf{E}\Delta_{AB}$  implies

$$\Pr\left(\Delta_{AB} \geq \frac{AB^2}{n^2}(1+17\eta)\right) \leq \Pr(\Delta_{AB} \geq (1+\eta)\mathbf{E}\Delta_{AB}) \leq e^{-\eta^2 c^2 n/8} \leq e^{-\alpha^2 c^2 n/2}. \quad (28)$$

The second inequality above follows from (5) applied with  $\epsilon = \eta$ , and the fact that  $\mathbf{E}\Delta_{AB} \geq (1/4)c^2 n$  (from (20) with  $A$  and  $B$  swapped). The last inequality in (28) follows from (22).

**Lower bound on  $\Delta$ .** Using (21) and (28), we get the following lower bound on the increase of the  $A$ -vote:

$$\begin{aligned} \Pr\left(\Delta = \Delta_{BA} - \Delta_{AB} \leq \frac{A^2 B}{n^2}(1-3\eta) - \frac{AB^2}{n^2}(1+17\eta)\right) \\ \leq \Pr\left(\left(\Delta_{BA} \leq \frac{A^2 B}{n^2}(1-3\eta)\right) \text{ or } \left(\Delta_{AB} \geq \frac{AB^2}{n^2}(1+17\eta)\right)\right) \leq e^{-\Theta(\alpha^2 c^2 n)}. \end{aligned}$$

Hence with probability at least  $1 - e^{-\Theta(\alpha^2 c^2 n)}$ ,

$$\begin{aligned} \Delta &\geq \frac{AB}{n} \left( \frac{A}{n}(1-3\eta) - \frac{B}{n}(1+17\eta) \right) \\ &\geq \frac{AB}{n} \left( \frac{A-B}{n} - 10\eta \right) \geq \frac{AB(A-B)}{n^2} - 3\eta n. \end{aligned}$$

Therefore, with probability at least  $1 - Te^{-\Theta(\alpha^2 c^2 n)} = 1 - e^{-\Theta(\alpha^2 c^2 n)}$ , for all steps  $t = 1, 2, \dots, T = O(\log n)$  when the size  $B_t$  of the  $B$ -vote is at least  $cn$ , we have

$$A_{t+1} = A_t + \Delta_t \geq A_t + \frac{A_t B_t (A_t - B_t)}{n^2} - 3\eta_t n. \quad (29)$$

In (29), substitute  $\eta_t = \alpha n / \sqrt{A_t B_t}$ ,  $A_t = n(1 + \nu_t)/2$  and  $B_t = n(1 - \nu_t)/2$  to get

$$\nu_{t+1} \geq \nu_t + \frac{1}{2}\nu_t(1 - \nu_t^2) - 12\frac{\alpha}{\sqrt{1 - \nu_t^2}}. \quad (30)$$

Thus, while  $K\alpha \leq \nu_t \leq 1/2$ , we have

$$\nu_{t+1} \geq \nu_t + \frac{3}{8}\nu_t - 15\alpha \geq \frac{5}{4}\nu_t.$$

This means that the number of steps required to increase the vote imbalance from  $\nu_0$  to  $1/2$  is at most  $\left\lceil \log_{5/4}(1/(2\nu_0)) \right\rceil$ .

For  $1/2 \leq \nu_t \leq 1 - 2c$ , we set  $\delta_t = 1 - \nu_t$  and (30) becomes

$$\delta_{t+1} \leq \delta_t - \frac{1}{2}(1 - \delta_t)\delta_t(2 - \delta_t) + 12\frac{\alpha}{\sqrt{\delta_t(2 - \delta_t)}}. \quad (31)$$

Since  $2c \leq \delta_t \leq 1/2$  and  $\alpha \leq c^{3/2}/36$ ,

$$\delta_{t+1} \leq \delta_t - \frac{3}{8}\delta_t + 9\frac{\alpha}{\sqrt{c}} \leq \frac{3}{4}\delta_t. \quad (32)$$

Thus the number of steps required to increase the vote imbalance from  $1/2$  to  $1 - 2c$  (that is, decrease  $\delta_t$  from  $1/2$  to  $2c$ ) is at most  $\left\lceil \log_{4/3}(1/(4c)) \right\rceil$ .



## 5 Phase II of analysis: $\omega \leq B \leq cn$

### 5.1 The main lemma and its application to expanders and random graphs

The analysis of this middle phase needs the property that small sets of vertices do not induce many edges, which holds for expanders and random regular graphs. The main Lemma 5 shows that for a graph with such a property, if the minority vote is still substantial, then one voting step reduces this minority by a constant factor with high probability. This implies that with high probability the minority vote reduces from  $cn$  (where  $c$  is a small constant) to  $\omega$  within  $O(\log n)$  steps (Corollary 4).

**Lemma 5** *Let  $G$  be a  $d$ -regular  $n$ -vertex graph with  $A$  and  $B$  votes, where  $A > B$ . Let  $0 < \alpha \leq 3/10$  and  $\gamma = \gamma(\alpha) = (1/2)(1 - 2\alpha)(1 - 3\alpha) > 0$ . If the set  $B$  is such that every superset  $S \supseteq B$  of size at most  $(1 + 1/\alpha)B$  spans at most  $\alpha dS$  edges (that is,  $|E(S)| \leq \alpha dS$ ), then one voting step reduces  $B$  by at least a factor  $1 - \gamma$ , with probability at least  $1 - e^{-\tilde{\gamma}B}$ , where  $\tilde{\gamma}$  is some constant bounded away from 0.*

**Corollary 4** *Let  $G$  be a  $d$ -regular  $n$ -vertex graph, and let  $0 < g < 1$  be such that for each subset of vertices  $S$  of size at most  $gn$ ,  $|E(S)| \leq (3/10)dS$ . Then the minority vote  $B$  is reduced from  $(3/13)gn$  to at most  $\omega$  within  $O(\log n)$  steps with probability at least  $1 - e^{-\Theta(\omega)}$ .*

*Proof.* For each step, apply Lemma 5 with  $\alpha = 3/10$  and  $\gamma = \gamma(3/10) > 0$ . In each step (by induction)  $B \leq (3/13)gn$ , so each superset  $S$  of  $B$  of size at most  $(1 + 1/\alpha)B$  has size at most  $(13/3)B \leq gn$ , implying  $|E(S)| \leq (3/10)dS = \alpha dS$ . Thus Lemma 5 implies that each step reduces  $B$  by a factor  $1 - \gamma < 1$ , with probability at least  $1 - e^{-\tilde{\gamma}B}$ .

Let  $r = \lceil \log(cn/\omega) / \log(1/(1 - \gamma)) \rceil = O(\log n)$ , so that  $\omega < (1 - \gamma)^{r-1}cn$  but  $\omega \geq (1 - \gamma)^r cn$ . The initial size of the minority vote  $B$  is  $B_0 \leq cn$ . We say that step  $i$  is successful, if the size of the  $B$  vote at the end of this step is  $B_i \leq (1 - \gamma)^i cn$ . If the steps  $1, 2, \dots, i - 1$  are successful, then  $B_{i-1} \leq (1 - \gamma)^{i-1} cn$  and the probability that  $B_i \leq (1 - \gamma)^i cn$  (that is, the probability that step  $i$  is successful) is at least the probability that a  $B$  vote of size  $(1 - \gamma)^{i-1} cn$  reduces in one step to  $(1 - \gamma)^i cn$ . This (conditional) probability is at least  $1 - \exp\{-\tilde{\gamma}(1 - \gamma)^{i-1} cn\}$  (Lemma 5). Therefore

$$\begin{aligned} \Pr(B_r \leq \omega) &\geq \Pr(\text{all steps } 1, 2, \dots, r \text{ are successful}) \\ &= \prod_{i=1}^r \Pr(\text{step } i \text{ is successful} \mid \text{steps } 1, 2, \dots, i-1 \text{ are successful}) \\ &\geq \prod_{i=1}^r (1 - \exp\{-\tilde{\gamma}(1 - \gamma)^{i-1} cn\}) \end{aligned} \tag{33}$$

$$= 1 - e^{-\Theta(\omega)}. \tag{34}$$

Thus with probability at least  $1 - e^{-\Theta(\omega)}$ ,  $B$  is reduced from  $cn$  to  $\omega$  in  $O(\log n)$  steps.  $\square$

**Lemma 6** *Let  $G$  be a  $d$ -regular  $n$ -vertex graph with  $\lambda = \lambda_G < 3/5$ . Then the minority vote  $B$  reduces from  $(3/13)(3/5 - \lambda)n$  to  $\omega$  within  $O(\log n)$  steps with probability at least  $1 - e^{-\Theta(\omega)}$ .*

*Proof.* This lemma follows from Corollary 4 applied with  $c = 1 - (2/5)(1 - \lambda)^{-1} > 0$ , after checking that  $|E(S)| \leq (3/10)dS$  whenever  $S \leq cn$ . It is shown in [20] that the conductance of graph  $G = (V, E)$  defined as

$$\Phi_G = \min_{\emptyset \neq S \subset V} \frac{nE(S, \bar{S})}{dS\bar{S}},$$

is at least  $1 - \lambda$ . This implies that  $E(S, \bar{S}) \geq (1 - \lambda)dS\bar{S}/n$ , so for any  $S \subseteq V$  of size at most  $cn$ ,

$$\begin{aligned} |E(S)| &= \frac{1}{2} (dS - E(S, \bar{S})) \leq \frac{1}{2} dS (1 - (1 - \lambda)\bar{S}/n) \\ &\leq \frac{1}{2} dS (1 - (1 - \lambda)(1 - c)) = \frac{3}{10} dS. \end{aligned}$$

□

As mentioned before, for constant  $d$  a random  $d$ -regular graph has eigenvalue  $\lambda_G \approx 2/\sqrt{d}$ , w.h.p. Thus, for constant  $d$ , the result which we have obtained for expanders (Lemma 6) applies to random regular graphs as well, provided that  $d$  is sufficiently large to guarantee  $\lambda_G < 3/5$ . To consider random regular graphs with degree which may grow with the number of vertices, we show the following lemma. This is a stronger version of a result from [10] that w.h.p. for  $3 \leq d \leq cn$  no set of vertices of size  $|S| \leq n/70$  induces more than  $d|S|/12$  edges.

**Lemma 7** *Let  $600 \leq d \leq n/K$  for some large constant  $K$ , and let  $G = (V, E)$  be a random  $d$ -regular  $n$  vertex graph. Let  $\alpha = 1/12$  and consider the event*

$$\mathbf{Q} = \{ \exists S \subseteq V : |S| \leq n/15 \text{ and } S \text{ spans at least } \alpha d|S| \text{ edges} \}.$$

*Then  $\Pr(\mathbf{Q}) \leq n^{-\delta}$ , for some constant  $\delta > 0$ .*

**Lemma 8** *Let  $d > 10$  and  $G$  be a random  $d$ -regular  $n$ -vertex graph with votes  $A$  and  $B$ . There is a constant  $c > 0$  (independent of  $d$ ) such that the minority vote  $B$  reduces from  $cn$  to  $\omega$  within  $O(\log n)$  steps with probability at least  $1 - e^{-\Theta(\omega)} - o(1/n)$ .*

*Proof.* For  $11 \leq d < 600$  use Lemma 6: in this case, a random  $d$ -regular graph has  $\lambda_G \leq (2\sqrt{d-1} + \epsilon)/d < 3/5$ .

For  $600 \leq d \leq n/K$ , Lemma 7 implies that with probability at least  $1 - o(1/n)$ ,  $E(S) \leq (1/12)dS < (3/10)dS$ , for each subset of vertices  $S$  of size at most  $n/15$ . Thus, applying Corollary 4 with  $g = 1/15$ , we conclude that the minority vote  $B$  is reduced from  $(3/13)gn = (1/65)n$  to at most  $\omega$  within  $O(\log n)$  steps with probability at least  $(1 - o(1/n))(1 - e^{-\Theta(\omega)})$ . □

## 5.2 Proof of Lemma 5

Consider first the following special case. For each vertex  $v \in B$ ,  $d_v^A = (1 - 2\alpha)d$  (so  $|E(B)| = \beta dB$ ), and for each  $v \in A$ ,  $d_v^B$  is either 0 or  $\alpha d$ . Since  $\sum_{v \in A} d_v^B = \sum_{v \in B} d_v^A$ , then the number of vertices  $v$  in  $A$  with  $d_v^B = \alpha d$  is  $B(1 - 2\alpha)/\alpha$ , so in this case the expected increase of the  $A$  vote is equal to

$$\mathbf{E}\Delta = (1 - 2\alpha)^2 B - \alpha^2 B(1 - 2\alpha)/\alpha = (1 - 2\alpha)(1 - 3\alpha)B.$$

The proof of Lemma 5 is based on confirming that this is the worst case, that is, we always have

$$\mathbf{E}\Delta \geq (1 - 2\alpha)(1 - 3\alpha)B. \quad (35)$$

Let  $|E(B)| = \beta dB$ . Since we must have  $|E(B)| \leq \alpha dB$ , then  $0 \leq \beta \leq \alpha$ . We have

$$\sum_{v \in A} d_v^B = \sum_{v \in B} d_v^A = dB - 2|E(B)| = dB(1 - 2\beta),$$

so

$$\mathbf{E}\Delta_{BA} = \sum_{v \in B} \left( \frac{d_v^A}{d} \right)^2 \geq \frac{1}{Bd^2} \left( \sum_{v \in B} d_v^A \right)^2 = \frac{1}{Bd^2} (dB - 2|E(B)|)^2 = B(1 - 2\beta)^2. \quad (36)$$

To bound  $\mathbf{E}\Delta_{AB}$ , we define  $C = \{v \in A : d_v^B > \alpha d\}$ . We have

$$\alpha dC \leq |E(C, B)| \leq dB,$$

so  $C + B \leq (1 + 1/\alpha)B$ , and the assumptions of the lemma imply that

$$|E(C \cup B)| \leq \alpha d(C + B). \quad (37)$$

For  $v \in C$ , we write  $d_v^B$  as a linear combination of  $d$  and  $\alpha d$ :

$$d_v^B = x_v d + y_v(\alpha d); \quad x_v + y_v = 1; \quad x_v, y_v \geq 0.$$

For  $v \in A \setminus C$ , we define  $y_v = d_v^B/(\alpha d)$ , so  $0 \leq y_v \leq 1$ , and set  $x_v = 0$ . We also define  $X = \sum_{v \in A} x_v$ ,  $Y = \sum_{v \in A} y_v$ , and  $Y_C = \sum_{v \in C} y_v$ . Using this definitions,

$$dB(1 - 2\beta) = \sum_{v \in A} d_v^B = dX + \alpha dY,$$

so

$$\alpha Y = B(1 - 2\beta) - X. \quad (38)$$

Furthermore,

$$|E(C \cup B)| = \sum_{v \in C} d_v^B + \beta dB = dX + \alpha dY_C + \beta dB, \quad (39)$$

and

$$C = X + Y_C. \quad (40)$$

Using (39) and (40) in (37), we get

$$X + \alpha Y_C + \beta B \leq \alpha(X + Y_C + B),$$

so

$$X \leq \frac{\alpha - \beta}{1 - \alpha} B. \quad (41)$$

We use (38) and (41) in bounding  $\mathbf{E}\Delta_{AB}$ :

$$\begin{aligned} \mathbf{E}\Delta_{AB} &= \sum_{v \in A} \left( \frac{d_v^B}{d} \right)^2 = \sum_{v \in C} (x_v + \alpha y_v)^2 + \sum_{v \in A \setminus C} (\alpha y_v)^2 \\ &\leq \sum_{v \in C} (x_v + \alpha^2 y_v) + \sum_{v \in A \setminus C} \alpha^2 y_v \\ &= X + \alpha^2 Y = X + \alpha(B(1 - 2\beta) - X) = X(1 - \alpha) + B\alpha(1 - 2\beta) \\ &\leq B(\alpha - \beta) + B\alpha(1 - 2\beta) = B(2\alpha - \beta - 2\alpha\beta). \end{aligned} \quad (42)$$

The bounds (36) and (42) give

$$\mathbf{E}\Delta = \mathbf{E}\Delta_{BA} - \mathbf{E}\Delta_{AB} \geq B((1 - 2\beta)^2 - 2\alpha + \beta + 2\alpha\beta) = B \cdot f_\alpha(\beta),$$

where

$$f_\alpha(\beta) = 4\beta^2 - (3 - 2\alpha)\beta + (1 - 2\alpha).$$

We check that  $f'_\alpha(\beta) = 8\beta - 3 + 2\alpha \leq 10\alpha - 3 \leq 0$ , for  $0 \leq \beta \leq \alpha \leq 3/10$ , so the minimum value of  $f_\alpha(\beta)$  for  $0 \leq \beta \leq \alpha$  is  $f_\alpha(\alpha) = (1 - 2\alpha)(1 - 3\alpha) = 2\gamma$ , and the bound (35) holds.

Thus  $\mathbf{E}\Delta \geq 2\gamma B$  and we show now that  $\Delta$  is at least  $\gamma B$  with high probability, by showing that w.h.p.  $\Delta_{AB}$  and  $\Delta_{BA}$  do not deviate from their expectations by more than  $\gamma B/2$ . For  $\Delta_{BA}$ , using (4) with  $\epsilon = \epsilon_1 = \gamma/(2(1 - 2\beta)^2) < 1$ , we obtain

$$\begin{aligned} \Pr\left(\Delta_{BA} \leq (1 - 2\beta)^2 B - \frac{\gamma}{2} B\right) &= \Pr\left(\Delta_{BA} \leq \left(1 - \frac{\gamma}{2(1 - 2\beta)^2}\right) (1 - 2\beta)^2 B\right) \\ &= \Pr(\Delta_{BA} \leq (1 - \epsilon_1) (1 - 2\beta)^2 B) \leq e^{-\gamma' B}, \end{aligned} \quad (43)$$

where  $\gamma' = (\gamma/(2(1 - 2\beta)))^2/3 > 0$  is a constant.

For  $\Delta_{AB}$ ,  $\epsilon = \epsilon_2 = \gamma/(2(2\alpha - \beta - 2\alpha\beta))$ . Now we cannot guarantee that  $\epsilon < 1$ . As long as  $(1 - 2\alpha)(1 - 3\alpha) < 4(2\alpha - \beta - 2\alpha\beta)$ , we apply (5) and obtain

$$\begin{aligned} \Pr\left(\Delta_{AB} \geq (2\alpha - \beta - 2\alpha\beta)B + \frac{\gamma}{2} B\right) &= \Pr(\Delta_{AB} \geq (1 + \epsilon_2)(2\alpha - \beta - 2\alpha\beta)B) \\ &\leq \exp\left\{-\frac{(1 - 2\alpha)^2(1 - 3\alpha)^2}{16(2\alpha - \beta - 2\alpha\beta)} \cdot \frac{B}{2}\right\} = e^{-\gamma'' B}, \end{aligned}$$

where  $\gamma'' = \frac{(1 - 2\alpha)^2(1 - 3\alpha)^2}{32(2\alpha - \beta - 2\alpha\beta)} > 0$  is a constant. If now  $\epsilon \geq 1$ , we apply (6), and obtain

$$\begin{aligned} \Pr\left(\Delta_{AB} \geq (2\alpha - \beta - 2\alpha\beta)B + \frac{\gamma}{2} B\right) &= \Pr(\Delta_{AB} \geq (1 + \epsilon_2)(2\alpha - \beta - 2\alpha\beta)B) \\ &\leq \left(\frac{e^\epsilon}{(1 + \epsilon)^{1 + \epsilon}}\right)^{(2\alpha - \beta - 2\alpha\beta)B} = e^{-\gamma''' B}, \end{aligned}$$

where  $\gamma''' = \ln((1 + \epsilon)^{(1 + \epsilon)/\epsilon}/e)(1 - 2\alpha)(1 - 3\alpha)/4$ .

The bounds above imply that

$$\begin{aligned} \Pr(\Delta \leq \gamma B) &\leq \Pr\left(\Delta_{BA} - \Delta_{AB} \leq ((1 - 2\beta)^2 B - \frac{\gamma}{2} B) - ((2\alpha - \beta - 2\alpha\beta)B + \frac{\gamma}{2} B)\right) \\ &\leq \Pr\left(\Delta_{BA} \leq (1 - 2\beta)^2 B - \frac{\gamma}{2} B\right) + \Pr\left(\Delta_{AB} \geq (2\alpha - \beta - 2\alpha\beta)B + \frac{\gamma}{2} B\right) \leq e^{-\tilde{\gamma} B}. \end{aligned}$$

### 5.3 Proof of Lemma 7

Let  $S$  also denote the size of set  $S$ . If  $S \leq 2\alpha d$ , then the set  $S$  has  $\binom{s}{2} < \alpha d S$  slots for edges, so it is not possible to insert  $\alpha d S$  edges into  $S$ . We therefore assume that  $2\alpha d + 1 \leq S \leq n/15$ .

The proof for this case is in the configuration model. Let  $\rho(S, k)$  be the probability a set size  $S$  spans at least  $k$  edges. Then, using the notation  $m!! = m!/(2^{m/2}(m/2)!)$ ,

$$\rho(S, k) = \binom{dS}{2k} 2k!! \frac{(dn - 2k)!!}{dn!!}.$$

After some simplification, we have that

$$\rho(S, k) = O(1) \frac{(dS)^{dS}}{k^k 2^k (dS - 2k)^{dS - 2k}} \frac{(dn - 2k)^{(dn - 2k)/2}}{dn^{dn/2}}.$$

To simplify further, let  $k = \alpha dS$ , and  $S = sn$ . Then

$$\begin{aligned}\rho(S, k) &= O(1) \left( \left( \frac{s}{2\alpha} \right)^{\alpha s} \frac{(1 - 2\alpha s)^{1/2 - \alpha s}}{(1 - 2\alpha)^{s(1 - 2\alpha)}} \right)^{dn} \\ &= O(1) [f(s, \alpha)]^{dn}.\end{aligned}$$

Thus, the probability that a random  $d$  regular  $n$ -vertex graph has the property  $\mathbf{Q}$  is

$$\begin{aligned}\Pr(\mathbf{Q}) &\leq O(1) \cdot e^{20d^2} \cdot \sum_{S=d/6}^{n/15} \binom{n}{S} [f(S/n, 1/12)]^{dn} \\ &= O(1) \cdot e^{20d^2} \cdot \sum \left( \frac{1}{s^s (1-s)^{1-s}} \left[ (1-s/6)^{1/2} \left( \frac{36s}{6-s} \left( \frac{6}{5} \right)^{10} \right)^{s/12} \right]^d \right)^n \\ &= O(1) \cdot e^{20d^2} \cdot \sum [F(s)]^n.\end{aligned}$$

The term  $e^{20d^2}$  takes into account that we only consider simple graphs.  $F(x)$  can be written as

$$F(x) = \frac{1}{x^x (1-x)^{1-x}} \left( \frac{6}{5} \right)^{5dx/6} \frac{6^{dx/6}}{6^{d/2}} [(6-x)^{6-x} x^x]^{d/12}.$$

The second derivative of  $\log F(x)$  is

$$\frac{\partial^2}{\partial x^2} \log F(x) = \frac{1}{x} \left( \frac{d}{12} - 1 \right) + \frac{d}{12} \frac{1}{6-x} - \frac{1}{1-x}.$$

Provided  $d \geq 132$ , this is strictly greater than zero for all  $x \in (0, 1/2)$ . Thus  $\log F(x)$  is convex, and is either monotone increasing, monotone decreasing, or has a unique minimum in the range  $x \in [d/6n, 1/15]$ . To find the maximum of  $F(x)$  it suffices to evaluate the function at  $x = d/6n$  and  $x = 1/15$ . Thus, assuming  $d \leq n/K$  for a large constant  $K$ ,

$$\Pr(\mathbf{Q}) \leq O(n) \max([F(d/6n)]^n, [F(1/15)]^n) \cdot e^{20d^2}.$$

However  $F(1/15) = ((0.999514)^d / 0.782759)$ , which gives  $F(1/15) = 0.954335$  when  $d = 600$ . Thus for  $d \geq 600$  we have  $[F(1/15)]^n \leq e^{-\Theta(dn)}$ . To bound  $[F(d/6n)]^n$ , observe that  $F(x) \leq (cx)^{x(d/12-1)}$ , for some constant  $c > 0$  and for all  $x \in (0, 1/2)$ . Hence, for a positive  $d \leq n/K$ , we have  $[F(d/6n)]^n \leq e^{-\Theta(d^2 \log(n/d))}$ , so for  $600 \leq d \leq n/K$ ,

$$\Pr(\mathbf{Q}) \leq e^{-\Theta(d^2 \log(n/d))} \leq n^{-\delta},$$

where  $\delta > 0$  is a constant.

## 6 Phase III of analysis: $1 \leq B \leq \omega$

**Lemma 9** *Let  $\omega = \omega(1)$  grow with  $n$  and  $\omega = o(n)$ . Let  $G$  be a  $d$ -regular  $n$ -vertex graph such that for each subset of vertices  $S$  of size at most  $(13/3)\omega$ ,  $|E(S)| \leq (3/10)dS$ . Then the minority vote  $B$  is reduced from  $\omega$  to 0 within  $O(\omega \log \omega)$  steps with probability at least  $1 - e^{-\Theta(\omega)}$ .*

*Proof.* Lemma 5 implies that if  $B = \omega$ , then the probability that the next step increases  $B$  above  $\omega$  is less than  $e^{-\tilde{\gamma}\omega}$ , where  $\tilde{\gamma} > 0$ . This also implies that if  $B < \omega$ , then the probability that the next step increases  $B$  above  $\omega$  is also less than  $e^{-\tilde{\gamma}\omega}$  (smaller  $B$  means smaller probability that one step will increase this  $B$  above  $\omega$ ). Therefore, the probability that  $B$  increases above  $\omega$  in any of the  $T = O(\omega \log \omega)$  steps is at most  $T e^{-\tilde{\gamma}\omega} = e^{-\Theta(\omega)}$ .

Let  $r = \lfloor \log \omega / \log(1/(1 - \gamma)) \rfloor + 1 = O(\log \omega)$ , so that  $(1 - \gamma)^{r-1}\omega \geq 1$  but  $(1 - \gamma)^r\omega < 1$ . The initial size of the minority vote  $B$  is  $B_0 \leq \omega$ . We say that step  $i$  is successful, if the size of the  $B$  vote at the end of this step is  $B_i \leq (1 - \gamma)^i\omega$ . If the steps  $1, 2, \dots, i - 1$  are successful, then  $B_{i-1} \leq (1 - \gamma)^{i-1}\omega$  and the probability that  $B_i \leq (1 - \gamma)^i\omega$  (that is, the probability that step  $i$  is successful) is at least the probability that a  $B$  vote of size  $(1 - \gamma)^{i-1}\omega$  reduces in one step to  $(1 - \gamma)^i\omega$ , which is at least  $1 - \exp\{-\tilde{\gamma}(1 - \gamma)^{i-1}\omega\}$  (Lemma 5). Therefore

$$\begin{aligned} \Pr(B_r = 0) &\geq \Pr(\text{all steps } 1, 2, \dots, r \text{ are successful}) \\ &\geq \prod_{i=1}^r (1 - \exp\{-\tilde{\gamma}(1 - \gamma)^{i-1}\omega\}) = p > 0, \end{aligned} \quad (44)$$

where  $p$  is a positive constant. Thus  $B$  is reduced from  $\omega$  to 0 within  $r = O(\log \omega)$  steps with constant (positive) probability. Consider now a sequence of  $\omega r = O(\omega \log \omega)$  steps, viewed as  $\omega$  phases, each consisting of  $r$  steps. If this sequence of steps does not reduce  $B$  from  $B_0 \leq \omega$  to 0, then there is a step which increases  $B$  above  $\omega$  or each phase starts with  $B \leq \omega$  but fails to reduce  $B$  to 0. This means that the probability that  $\omega r$  steps do not reduce  $B_0 \leq \omega$  to 0 is at most  $e^{-\Theta(\omega)} + (1 - p)^\omega = e^{-\Theta(\omega)}$ .  $\square$

**Corollary 5** *Let  $\omega = \omega(1)$  grow with  $n$  and  $\omega = o(n)$ . If  $G = (V, E)$  is a  $d$ -regular expander with  $\lambda_G < 3/5$  or it is a random  $d$ -regular graph with  $d > 10$ , then voting reduces  $B$  from at most  $\omega$  to 0 in  $O(\omega \log \omega)$  steps with probability at least  $1 - e^{-\Theta(\omega)}$ .*

*Proof.* If  $G$  is a  $d$ -regular expander with  $\lambda_G \leq 3/5$ , then the assumptions of Lemma 9 are fulfilled for  $G$ , as shown in the proof of Lemma 6. If  $G$  is a random  $d$ -regular graph with  $d = \omega(1)$ , then the assumptions of Lemma 9 are also fulfilled for  $G$  according to Lemma 7. If  $d > 10$  but  $d = O(1)$ , then  $G$  has eigenvalue  $\lambda_G < 3/5$ , w.h.p. [17].  $\square$

## 7 Putting the phases together

To conclude the proof of our main Theorems 1 and 2, it remains to check how the three phases fit together. For expanders (Theorem 2), first use Corollary 2 with  $c = 1/10$  to get constant  $K = K(c)$  such that if the initial imbalance of vote is  $\nu_0 \geq K\lambda_G$ , then the minority vote reduces to  $n/10$  within  $O(\log(1/\nu_0))$  steps. Then use Lemma 6 with  $\omega = \log n / \log \log n$  and assume that  $\lambda_G \leq 1/6$  to show that the minority vote reduces from  $n/10$  to  $\omega$  in  $O(\log n)$  steps. Finally, apply Corollary 5 with the same  $\omega$  to show that the minority vote decreases from  $\omega$  to 0 in  $O(\log n)$  steps.

For the random regular graphs, Lemma 8 gives the constant  $c < 1/2$  which defines the beginning of phase II. Then Corollary 3 can be used to find the constant  $K$  for Theorem 1. The transition from phase II to phase III is at the same  $\omega = \log n / \log \log n$  as before.

According to our analysis, we can also derive the following corollary.

**Corollary 6** *Assume an adversary can change the opinion of at most  $f = o(\nu_0 n)$  vertices during the execution of the algorithm. Then, under the assumptions of Theorems 1 and 2 all but  $O(f)$  vertices will adopt opinion  $A$  within  $O(\log n)$  steps, w.h.p.*

To obtain the statement, observe that the assumptions in Phases I, II, and III are fulfilled w.r.t.  $A - f$  and  $B + f$  as long as  $B \geq C' \cdot f$ , where  $C'$  is a suitable large constant.



## References

- [1] M. Abdullah and M. Draief. *Consensus on the Initial Global Majority by Local Majority Polling for a Class of Sparse Graphs*. (2013) [www.arXiv.org](http://www.arXiv.org)
- [2] D. Aldous and J. Fill. *Reversible Markov Chains and Random Walks on Graphs*, <http://stat-www.berkeley.edu/pub/users/aldous/RWG/book.html>.
- [3] N. Alon and F. R. K. Chung. *Explicit construction of linear sized tolerant networks*. Discrete Math., 72:15-19, (1989).
- [4] L. Becchetti, A. Clementi, E. Natale, F. Pasquale, R. Silvestri, L. Trevisan. *Simple Dynamics for Majority Consensus*. (2013) [www.arXiv.org](http://www.arXiv.org)
- [5] B. Bollobás. *The isoperimetric number of random regular graphs*. Europ. J. Combinatorics, 9:241-244, (1988).
- [6] S. Brahma, S. Macharla, S. P. Pal, S. R. Singh. Fair Leader Election by Randomized Voting. In *ICDCIT 2004*, pages 22-31, 2004.
- [7] C. Cooper, R. Elsässer, H. Ono, T. Radzik. Coalescing Random Walks and Voting on Graphs. In *PODC 2012*, pages 47-56, 2012.
- [8] C. Cooper, R. Elsässer, H. Ono, T. Radzik. *Coalescing Random Walks and Voting on Connected Graphs*. SIAM J. Discrete Math. 27(4):1748-1758, (2013).
- [9] C. Cooper, A. Frieze, B. Radzik. *Multiple Random Walks in Random Regular Graphs*. SIAM J. Discrete Math. 23(4):1738-1761, (2009).
- [10] C. Cooper, A. Frieze, B. Reed. *Random regular graphs of non-constant degree: connectivity and Hamilton cycles*. Combinatorics Prob. & Comp. 11:249-262, (2002).
- [11] J. Cruise and A. Ganesh, *Probabilistic consensus via polling and majority rules*. (2013) [www.arXiv.org](http://www.arXiv.org).
- [12] X. Deng and C. Papadimitriou. *On the Complexity of Cooperative Solution Concepts*. Mathematics of Operations Research 19(2):257-266, (1994).
- [13] B. Doerr, L.A. Goldberg, L. Minder, T. Sauerwald, C. Scheideler: Stabilizing Consensus with the Power of Two Choices. In *SPAA 2011* pages 149-158, 2011.
- [14] P. Donnelly and D. Welsh. *Finite particle systems and infection models*. Math. Proc. Camb. Phil. Soc. 94(1):167-182, (1983).
- [15] N. Fountoulakis and K. Panagiotou. *Rumor Spreading on Random Regular Graphs and Expanders*. APPROX and RANDOM 2010, pages 560-573, 2010.
- [16] A. Frieze and T. Łuczak. *On the independence and chromatic numbers of random graphs*. J. Combinatorial Theory, Ser. B, 54:123-132, (1992).
- [17] J. Friedman A proof of Alon's second eigenvalue conjecture. In *STOC 2003*, pages 720-724, 2003.
- [18] D. Gifford. Weighted Voting for Replicated Data. In *SOSP 1979*, pages 150-162, 1979.

- [19] Y. Hassin and D. Peleg. *Distributed probabilistic polling and applications to proportionate agreement*. Information & Computation, 171(2):248-268, (2001).
- [20] M. Jerrum and A. Sinclair. Conductance and the rapid mixing property for Markov chains: the approximation of permanent resolved. In *STOC 1988*, pages 235-244, 1988.
- [21] B. Johnson, *Design and Analysis of Fault Tolerant Digital Systems*, Addison-Wesley, (1989).
- [22] E. Mossel , J. Neeman, O. Tamuz, *Majority Dynamics and Aggregation of Information in Social Networks*. (2012) [www.arXiv.org](http://www.arXiv.org).
- [23] T. Nakata, H. Imahayashi, M. Yamashita. Probabilistic local majority voting for the agreement problem on finite graphs. In *COCOON 1999*, pages 330-338, 1999.
- [24] R.I. Oliviera. *On the Coalescence Time of Reversible Random Walks*. Trans. Amer. Math. Soc., 364:2109–2128, (2012).
- [25] N.C. Wormald. Models of random regular graphs. In *Surveys in Combinatorics* (J. D. Lamb and D. A. Preece, eds), pp. 239–298.